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Hahn-Banach extension theorem in generating spaces of quasi-norm family

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Abstract. In this paper, we give a constructive proof of Hahn-Banach extension theorem in finite dimensional generating spaces of quasi-norm family(G.S.Q-N.F). On the other hand we establish Hahn-Banach extension theorem on generating spaces of semi-norm family(G.S.S-N.F) and some consequences of the same theorem are studied.

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1. Introduction

It is well known that metric and norm structures play pivotal role in functional analysis. So in order to develop this one has to take care of the suitable fuzzification of these structures. Historically, the problem of generalization of the metric structure came first. Different authors introduced ideas of fuzzy-metric space([6], [13]), probabilistic metric spaces [12],quasi metric space, statistical metric space[12], soft inner product spaces[4] fuzzy normed linear space[1], fuzzy soft topological spaces[10], generalized open fuzzy set[11], 2-fuzzy inner product space[2]etc. S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung and S. M. Kang [3] first introduced a definition of generating spaces of quasi-metric family, which generalizes those of fuzzy metric spaces in the sense of Kaleva & Seikkala [6] and Menger probabilistic metric spaces [12]. They also proved several fixed point theorems in quasi-metric family. J. S. Jung, B. S. Lee and Y. J. Cho, [5] established some fixed point theorems in generating spaces of quasi-metric family. In 2006, Xiao & Zhu [14] introduced a concept of generating spaces of quasi-norm family (G.S.Q-N.F) and studied linear topological structures. They introduced the concept of convergent sequence, Cauchyness, completeness, compactness etc. and established some fixed point theorems specially Schauder-type fixed point theorem in such spaces. In [8], we have established some results in finite dimensional G.S.Q-N.F and derived a G.S.Q-N.F from a generalized B-S fuzzy normed [1] linear space. We have also introduced in [9], the idea of continuity, boundedness of linear operators and deduced quasi-norm family of bounded linear operators leading to the development of dual space.

In this paper, we give a constructive proof of Hahn-Banach extension theorem in finite dimensional generating spaces of quasi-norm family. On the other hand we establish Hahn-Banach extension theorem on generating spaces of semi-norm family and some consequences of the same theorem are studied.

The organization of the paper is as follows:
Section 1, comprises some preliminary results.
In section 2, we establish the Hahn-Banach extension theorem in finite dimensional G.S.Q-N.F.
In section 3, an idea of operator semi-norm family is introduced and Hahn-Banach extension theorem is proved in G.S.S-N.F.
Throughout this paper straightforward proofs are omitted.

2. Preliminaries

In this section some preliminary results are given which are related to this paper.

Definition 2.1 ([9]). Let X be a linear space over E(Real or Complex) and θ be the origin of X. Let

$$Q = \{|.|_\alpha : \alpha \in (0,1)\}$$

be a family of mappings from X into $[0,\infty)$. (X,Q) is called a generating space of quasi-norm family and Q, a quasi-norm family, if the following conditions are satisfied:
(QN1) $|x|_\alpha = 0 \ \ \forall \alpha \in (0,1)$ iff $x = \theta$;
(QN2) $|ex|_\alpha = |e||x|_\alpha \quad \forall\ x \in X, \forall \alpha \in (0,1)$ and $\forall\ e \in E$;
(QN3) for any $\alpha \in (0,1)$ there exists a $\beta \in (0,\alpha]$ such that
$|x+y|_\alpha \leq |x|_\beta + |y|_\beta$ for all $x, y \in X$;
(QN4) for any $x \in X$, $|x|_\alpha$ is non-increasing for $\alpha \in (0,1)$.
(X,Q) is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where
(QN-3u) for any $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that
$|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta$ for any $n \in Z^+$, $x_i \in X (i = 1,2,....,n)$;
(QN-3t) for any $\alpha \in (0,1]$ there exists a $\beta \in (0,\alpha]$ such that
$|x+y|_\alpha \leq |x|_\alpha + |y|_\beta$ for $x, y \in X$;
(QN-3e) for any $\alpha \in (0,1]$, it holds that $|x+y|_\alpha \leq |x|_\alpha + |y|_\alpha$ for $x, y \in X$.

Definition 2.2 ([9]). Let $T : (X_1, Q_1) \to (X_2, Q_2)$ be an operator. Then T is said to be bounded if corresponding to each $\alpha \in (0,1)$, $\exists\ M_\alpha > 0$ such that

$$|T(x)|^2_\alpha \leq\ M_\alpha |x|^1_{1-\alpha} \ \ \forall x \in\ X_1.$$

Definition 2.3 ([9]). Let (X_1, Q_1) and (X_2, Q_2) be two generating spaces of quasi-norm family and $\alpha \in (0,1)$. An operator $T : (X_1, Q_1) \to (X_2, Q_2)$ is said to be $\alpha-$level bounded if $\exists\ \ M_\alpha > 0$ such that $|T(x)|^2_\alpha \leq\ M_\alpha |x|^1_{1-\alpha} \ \ \forall x \in\ X_1$.

Theorem 2.4 ([9]). *Let (X_1, Q_1) and (X_2, Q_2) be two G.S.Q-N.F. We denote by $B(X_1, X_2)$ the set of all bounded linear operators from (X_1, Q_1) to (X_2, Q_2). Then $B(X_1, X_2)$ is also a linear space.*

Theorem 2.5 ([9]). *Let (X_1, Q_1) and (X_2, Q_2) be two G.S.Q-N.F where Q_1 satisfies* **(QN6)***: if $x(\neq \theta) \in X_1$ then $|x|^1_\alpha > 0\ \forall \alpha \in (0, 1)$. For $T \in B(X_1, X_2)$ and $\alpha \in (0, 1)$ we define*

$$|T|_\alpha = \bigvee_{x(\neq\theta)\in X_1} \frac{|T(x)|^2_\alpha}{|x|^1_{1-\alpha}}$$

Then $(B(X_1, X_2), Q)$ is a G.S.Q-N.F.

Note 2.6. *Let (X_1, Q_1) and (X_2, Q_2) be two G.S.Q-N.F where Q_1 satisfies* **(QN6)***. If T is an α-level bounded linear operator for some $\alpha \in (0, 1)$ then $|T|_\alpha$ exists.*

Definition 2.7 ([7]). Let X be a linear space and p be a function from X to R. Then p is said to be a sub-linear functional on X if the followings hold:
(i) $p(\lambda\, x) = |\lambda|\ p(x)$ for all $\lambda \in R$ and $\forall\, x \in X$;
(ii) $p(x + y) \leq p(x) + p(y) \quad \forall\, x,\ y \in X$.

Theorem 2.8 ([7]). *Let X be any linear space (Real or Complex) and p be a sub-linear functional on X. Let f be a linear functional which is defined on a subspace Z of X satisfying $|f(x)| \leq p(x) \quad \forall\, x \in Z$. Then f has a linear extension $\hat{f}$ from Z to X satisfying*
$|\hat{f}(x)| \leq p(x) \quad \forall\, x \in X$ and $\hat{f}(x) = f(x) \quad \forall\, x \in Z$.

Note 2.9 ([9]). *Let (X, Q) be a G.S.Q-N.F satisfying* **(QN6)***. If T is an α-level bounded linear functional on X for some $\alpha \in (0, 1)$ then T is continuous on X.*

3. Hahn-Banach extension theorem in finite dimensional G.S.Q-N.F

In this section we define a quasi sub-linear functional on a linear space X and establish the Hahn-Banach extension theorem in finite dimensional G.S.Q-N.F.

Definition 3.1. Let X be a linear space and $P = \{p_\alpha : \alpha \in (0, 1)\}$ be a family of functions from X to R. Then P is called a family of quasi sub-linear functional on X if the followings hold:
(i) $p_\alpha(\lambda\, x) = |\lambda|\ p_\alpha(x)$ for all $\lambda \in R$, $\forall\, x \in X$ and $\forall \alpha \in (0, 1)$;
(ii) for any $\alpha \in (0, 1)$ there exists a $\beta \in (0, \alpha]$ such that
$p_\alpha(x + y) \leq p_\beta(x) + p_\beta(y) \quad \forall\, x,\ y \in X$.

Theorem 3.2. *Let X be any finite dimensional vector space and $P = \{p_\alpha : \alpha \in (0, 1)\}$ be a family of quasi sub-linear functional on X. Let $\alpha \in (0, 1)$ and f be a linear functional which is defined on a subspace Z of X satisfying $|f(x)| \leq p_\alpha(x) \quad \forall\, x \in Z$. Then f has a linear extension $\hat{f}$ from Z to X satisfying*
$|\hat{f}(x)| \leq p_\beta(x) \quad \forall\, x \in X$ for some $\beta \in (0, \alpha]$ and $\hat{f}(x) = f(x) \quad \forall\, x \in Z$.

Proof. Let $\alpha \in (0, 1)$ and f be a linear functional which is defined on a subspace Z of X satisfying $|f(x)| \leq p_\alpha(x) \quad \forall\, x \in Z$.
If $Z = X$ then nothing to prove. Let $Z \neq X$, then there exists $x_0 \in X - Z$. Clearly

$x_0 \neq \theta$ and the space Z_1 generated by $Z \cup \{x_0\}$ is also a subspace of X and has higher dimension than Z.
Let $x, y \in Z$, then
$f(x) - f(y) = f(x-y) \leq p_\alpha(x-y) = p_\alpha(x + x_0 - x_0 - y)$
$\leq p_{\beta_1}(x+x_0) + p_{\beta_1}(x_0+y) \;\; \forall x_0 \in X,$ for some $\beta_1 \in (0,\alpha]$
$\Rightarrow f(x) - p_{\beta_1}(x+x_0) \leq f(y) + p_{\beta_1}(y+x_0) \;\; \forall x, y \in Z$
$\Rightarrow \bigvee_{x\in Z} \{f(x) - p_{\beta_1}(x+x_0)\} \leq \bigwedge_{y \in Z} \{f(y) + p_{\beta_1}(y+x_0)\}$
Let $\gamma \in R$ such that
$\bigvee_{x\in Z} \{f(x) - p_{\beta_1}(x+x_0)\} \leq \gamma \leq \bigwedge_{y\in Z} \{f(y) + p_{\beta_1}(y+x_0)\}$
Let $z \in Z_1$ then z is of the form $z = x + tx_0$, where $t \in R$ and $x \in Z$. Clearly this representation is unique. If we define
$f_1(z) = f(x) - t\gamma \;\; \forall y \in Z_1$, then f_1 will be a linear functional defined on Z_1 such that
$f_1(x) = f(x) \;\; \forall x \in Z.$
If $t > 0$ then
$f_1(z) = t\{f(\frac{x}{t}) - \gamma\} \leq tp_{\beta_1}(\frac{x}{t} + x_0) = p_{\beta_1}(x + tx_0) = p_{\beta_1}(z).$
If $t < 0$ then
$f(\frac{x}{t}) - \gamma \geq -p_{\beta_1}(\frac{x}{t} + x_0) = -\frac{1}{|t|}p_{\beta_1}(x+tx_0) = \frac{1}{t}p_{\beta_1}(z).$
Hence $f_1(z) \leq p_{\beta_1}(z)$.
If $t = 0$ then
$f_1(z) = f(z) \leq p_\alpha(z) \leq p_{\beta_1}(z).$
Now $-f_1(z) = f_1(-z) \leq p_{\beta_1}(-z) = |-1|p_{\beta_1}(z) = p_{\beta_1}(z).$
Hence $|f_1(z)| \leq p_{\beta_1}(z) \;\; \forall z \in Z_1$.
Since X is finite dimensional, after a finite number of steps we will get a linear extension f_n of f defined on $Z_n = X$ such that
$|f_n(z)| \leq p_{\beta_n}(z) \;\; \forall z \in X.$
We choose $f_n = \hat{f}$ and $\beta_n = \beta$, then the theorem follows. □

Remark 3.3. If there is a decreasing sequence $\{\alpha_n\}$ in $(0, \alpha)$ with $\lim_{n\to\infty} \alpha_n = \beta > 0$ and
$|x+y|_{\alpha_i} \leq |x|_{\alpha_{i+1}} + |y|_{\alpha_{i+1}}$ for all $x, y \in X$, then the Theorem 3.2 can be extended to a countably infinite dimensional space.

Theorem 3.4. *Let (X, Q) be a finite dimensional generating space of quasi-norm family satisfying (QN6) and f be a bounded linear functional which is defined on a subspace Z of X. Then f has a linear extension $\hat{f}$ from Z to X which is β-level bounded on X for some $\beta \in (0,1)$ and $|f|_\beta \leq |\hat{f}|_\beta \leq |f|_{1-\beta}$.*

Proof. Let $\alpha \in (0,1)$ and we define
$p_\alpha(x) = |f|_\alpha \, |x|_\alpha \;\; \forall x \in X.$
Then clearly $P = \{p_\alpha : \alpha \in (0,1)\}$ is a family of quasi sub-linear functional on X.
Let $\alpha_0 \in [0.5, 1)$ then
$|f(x)| \leq |f|_{\alpha_0} \, |x|_{1-\alpha_0} \leq |f|_{1-\alpha_0} \, |x|_{1-\alpha_0} \; \forall x \in Z;$
$\Rightarrow f(x) \leq p_{1-\alpha_0}(x) \;\; \forall x \in Z.$
So by Theorem 3.2, f has a linear extension $\hat{f}$ from Z to X satisfying

$|\hat{f}(x)| \leq\ p_{\beta_0}(x)\ \ \forall\, x \in\ X$, for some $\beta_0 \in (0, 1-\alpha_0)$.
$\Rightarrow\ |\hat{f}(x)|\ \leq\ |f|_{\beta_0}\ |x|_{\beta_0}\ \ \forall\, x \in\ X.$
Let $\beta = (1-\beta_0) \in (0,1)$ then
$|\hat{f}(x)|\ \leq\ |f|_{1-\beta}\ |x|_{1-\beta}\ \ \forall\, x \in\ X$..........(1)
So $\hat{f}$ is a β-level bounded linear functional on X and

$$|\hat{f}|_{\beta} = \bigvee_{x(\neq\theta)\in\ X} \frac{|\hat{f}(x)|}{|x|_{1-\beta}} \leq\ |f|_{1-\beta} \quad \text{by (1)}.$$

Since Z is a subspace of X and $\hat{f}(x) = f(x)\ \ \forall\, x \in\ Z$,

$$\bigvee_{x(\neq\theta)\in\ X} \frac{|\hat{f}(x)|}{|x|_{1-\beta}} \geq \bigvee_{x(\neq\theta)\in\ Z} \frac{|f(x)|}{|x|_{1-\beta}};$$

$\Rightarrow\ |\hat{f}|_{\beta} \geq\ |f|_{\beta}$.
Hence $|f|_{\beta}\ \leq\ |\hat{f}|_{\beta}\ \leq\ |f|_{1-\beta}$. □

Theorem 3.5. *Let $(X,\ Q)$ be a generating space of semi-norm family satisfying (QN6) and f be a bounded linear functional which is defined on a subspace Z of X. Then for each $\alpha \in\ (0,1)$, f has a linear extension $\hat{f_\alpha}$ from Z to X which is α-level bounded on X and $|f|_{\alpha}\ =\ |\hat{f_\alpha}|_{\alpha}$.*

Proof. Let $\alpha \in\ (0,1)$ and we define
$p_{\alpha}(x) = |f|_{\alpha}\ |x|_{1-\alpha}\ \ \ \forall\, x \in\ X.$
Then clearly p_{α} is a sub-linear functional on X and
$|f(x)| \leq\ |f|_{\alpha}\ |x|_{1-\alpha} = p_{\alpha}(x)\ \forall\, x \in\ Z.$
So by the Hahn-Banach theorem on linear space, f has a linear extension $\hat{f_\alpha}$ from Z to X satisfying
$|\hat{f_\alpha}(x)| \leq\ p_{\alpha}(x)\ \ \forall\, x \in\ X.$
$\Rightarrow\ |\hat{f_\alpha}(x)|\ \leq\ |f|_{\alpha}\ |x|_{1-\alpha}\ \ \forall\, x \in\ X.$
Hence $\hat{f_\alpha}$ is α-level bounded on X and
$|\hat{f_\alpha}|_{\alpha} \leq\ |f|_{\alpha}$.
Again from definition $|\hat{f_\alpha}|_{\alpha} \geq\ |f|_{\alpha}$.
Hence $|\hat{f_\alpha}|_{\alpha}\ =\ |f|_{\alpha}\ \forall \alpha \in\ (0,1)$. □

Application:

Let $(X,\ Q)$ be a finite dimensional generating space of quasi-norm family satisfying (QN6). Then there exists a nontrivial continuous linear functional defined on X.

Proof. Let $Dim X\ =\ n$ and $\{e_1,\ e_2, ...,\ e_n\}$ be a basis of X. Let Z be a subspace of X generated by $\{e_1\}$. Let us define a functional $f : Z \rightarrow\ R$ by $f(x) = \lambda$ if $x = \lambda\ e_1$. Then clearly f is a linear functional on Z.
Now $|f(x)| = |\lambda| = |\lambda| \frac{|e_1|_{1-\alpha}}{|e_1|_{1-\alpha}}$;
$\Rightarrow\ |f(x)| = \frac{1}{|e_1|_{1-\alpha}}\ |\lambda\ e_1|_{1-\alpha} = \frac{1}{|e_1|_{1-\alpha}}\ |x|_{1-\alpha}$;
$\Rightarrow\ |f(x)| = M_{\alpha}|x|_{1-\alpha}\ \ \ \forall\, x \in\ Z,\ \ \forall\, \alpha \in\ (0,\ 1)$, where $M_{\alpha} = \frac{1}{|e_1|_{1-\alpha}}$.
Hence f is a bounded linear functional on Z. By Theorem 3.5, f has a linear extension $\hat{f}$ from Z to X which is β-level bounded on X for some $\beta \in\ (0,1)$. Since $\hat{f}$ is β-level bounded on X, it is continuous on X. □

4. Redefined operators semi-norm family and Hahn-Banach extension theorem

In this section, we introduce a concept of operator semi-norm family and prove the Hahn-Banach extension theorem on generating spaces of quasi-norm family.

Theorem 4.1. *Let (X_1, Q_1) be a G.S.Q-N.F and (X_2, Q_2) be a generating space of semi-norm family (G.S.S-N.F) satisfying (QN6). For $T \in B(X_1, X_2)$ and $\alpha \in (0,1)$ we define*

$$|T|^s_\alpha = \bigvee_{x \in X_1, |x|^1_{1-\alpha} \leq 1} \{|T(x)|^2_\alpha\}$$

Then $(B(X_1, X_2), Q^s)$ is a G.S.S-N.F satisfying (QN6), where $Q^s = \{|.|^s_\alpha : \alpha \in (0,1)\}$.

Proof. Clearly $|T|^s_\alpha \geq 0 \ \ \forall \alpha \in (0,1)$ and the condition (QN2) is directly followed from definition.
For (QN1), if $T = O$ then $T(x) = \theta \ \forall x \in X_1$
$\Rightarrow |T|^s_\alpha = 0 \ \ \forall \alpha \in (0,1)$.
Conversely let $|T|^s_\alpha = 0 \ \ \forall \alpha \in (0,1)$. We have to prove $T = O$ i.e. $T(x) = \theta \ \forall x \in X_1$.
If possible let $x_0 \in X_1$ and $T(x_0) \neq \theta$. Since T is linear $x_0 \neq \theta$. Then by (QN1) there exists $\alpha_0 \in (0,1)$ such that $|x_0|^1_{1-\alpha_0} \neq 0$.
Let $y = \frac{x_0}{|x_0|^1_{1-\alpha_0}} \in X_1$.
Then $|T(y)|^2_{\alpha_0} = 0$
$\Rightarrow |T(x_0)|^2_{\alpha_0} = 0$. But $T(x_0) \neq \theta$, which contradicts the fact that (X_2, Q_2) satisfies (QN6).
Hence $T(x) = \theta \ \forall x \in X_1 \ \Rightarrow \ T = O$.

For (QN3e), let $T_1, \ T_2 \in B(X_1, X_2)$ and $\alpha \in (0,1)$ then

$$|T_1 + T_2|^s_\alpha = \bigvee_{x \in X_1, |x|^1_{1-\alpha} \leq 1} \{|(T_1 + T_2)(x)|^2_\alpha\}$$

$$\leq \bigvee_{x \in X_1, |x|^1_{1-\alpha} \leq 1} \{|T_1(x)|^2_\alpha\} + \bigvee_{x \in X_1, |x|^1_{1-\alpha} \leq 1} \{|T_2(x)|^2_\alpha\}$$

$= |T_1|^s_\alpha + |T_2|^s_\alpha$.
For (QN4), let $\alpha > \beta$ then,
$1 - \alpha < 1 - \beta \ \Rightarrow \ |x|^1_{1-\alpha} \geq \ |x|^1_{1-\beta}$

$$\Rightarrow \bigvee_{x \in X_1, |x|^1_{1-\alpha} \leq 1} \{|T_1(x)|^2_\alpha\} \leq \bigvee_{x \in X_1, |x|^1_{1-\beta} \leq 1} \{|T_1(x)|^2_\beta\}.$$

For (QN6), let $T \neq O$ then there exists a $z (\neq \theta) \in X_1$ such that $T(z) \neq \theta$.
Let $\alpha_0 \in (0, \ 1)$ then,

$$|T|^s_{\alpha_0} = \bigvee_{x \in X_1, |x|^1_{1-\alpha_0} \leq 1} \{|T(x)|^2_{\alpha_0}\} \ > 0 \text{ if } |z|^1_{1-\alpha_0} \leq \ 1.$$

If $|z|^1_{1-\alpha_0} \ > \ 1$, let $z_0 = \frac{z}{|z|^1_{1-\alpha_0}} \in X_1$, then $T(z_0) \neq \theta$ and hence $|T|^s_{\alpha_0} \ > 0$.
Hence $(B(X_1, X_2), Q^s)$ is a G.S.S-N.F satisfying (QN6). □

Note 4.2. *Let (X_1, Q_1) be a G.S.Q-N.F and (X_2, Q_2) be a generating space of semi-norm family (G.S.S-N.F) satisfying (QN6). Then for each $\alpha \in (0,1)$, $|.|^s_\alpha$ is a norm on $B(X_1, \ X_2)$.*

Definition 4.3. Let $(X,\ Q)$ be a generating space of semi-norm family (G.S.S-N.F) satisfying (QN6). Then $(X,\ Q)$ is said to be a generating space of norm family (G.S.N.F).

Note 4.4. *Let (X_1, Q_1) and (X_2, Q_2) be two G.S.Q-N.F where Q_1 satisfies* **(QN6)**. *For $T \in B(X_1, X_2)$ and $\alpha \in (0,1)$ we define*
$|T|_\alpha = \bigvee_{x(\neq\theta)\in X_1} \dfrac{|T(x)|^2_\alpha}{|x|^1_{1-\alpha}}$
and
$|T|^s_\alpha = \bigvee_{x\in X_1, |x|^1_{1-\alpha}\leq 1} \{|T(x)|^2_\alpha\}.$
Then $|T|_\alpha = |T|^s_\alpha \ \ \forall \alpha \in (0,1)$.

Proof. If possible let $|T|_{\alpha_0} > |T|^s_{\alpha_0}$ for some $\alpha_0 \in (0,1)$. Then there exists an element $x_0 \in X_1$ such that
$\frac{|T(x_0)|^2_{\alpha_0}}{|x_0|^1_{1-\alpha_0}} > |T|^s_{\alpha_0}$.
Let $x = \frac{x_0}{|x_0|^1_{1-\alpha_0}}$. Then $|x|^1_{1-\alpha_0} = 1$ and $|T(x)|^2_{\alpha_0} > |T|^s_{\alpha_0}$, which is a contradiction because an element can not greater than its supremum.
Conversely if $|T|^s_{\alpha_0} > |T|_{\alpha_0}$ for some $\alpha_0 \in (0,1)$. Then there exists an element $y_0 \in X_1$ such that
$|T(y_0)|^2_{\alpha_0} > |T|_{\alpha_0}$ where $|y_0|^1_{1-\alpha_0} \leq 1$.
Clearly $y_0 \neq \theta$. Let $y = \frac{y_0}{|y_0|^1_{1-\alpha_0}}$. Then
$\frac{|T(y)|^2_{\alpha_0}}{|y|^1_{1-\alpha_0}} = \frac{|T(y_0)|^2_{\alpha_0}}{|y_0|^1_{1-\alpha_0}} \geq |T(y_0)|^2_{\alpha_0} > |T|_{\alpha_0}$, which is a contradiction because an element can not greater than its supremum. Hence the theorem. □

Theorem 4.5. *Let (X_1, Q_1) and (X_2, Q_2) be two generating spaces of quasi-norm family (G.S.Q-N.F). If $T \in B(X_1, X_2)$, then*
$|T(x)|^2_\alpha \leq |T|^s_\alpha |x|^1_{1-\alpha} \ \ \forall \alpha \in (0,1),\ \forall\ x \in X_1$.

Proof. Let T be bounded, then corresponding to each $\alpha \in (0,1)$, $\exists\ M_\alpha > 0$ such that

$$|T(x)|^2_\alpha \leq M_\alpha |x|^1_{1-\alpha} \ \ \forall x \in X_1.$$

Let $\alpha \in (0,1)$.
Now if $|x|^1_{1-\alpha} = 0$, then $|T(x)|^2_\alpha = 0$. So $|T(x)|^2_\alpha \leq |T|^s_\alpha |x|^1_{1-\alpha}$ holds.
If $|x|^1_{1-\alpha} \neq 0$, then $|T(\frac{x}{|x|^1_{1-\alpha}})|^2_\alpha \leq |T|^s_\alpha$.
So $|T(x)|^2_\alpha \leq |T|^s_\alpha |x|^1_{1-\alpha}$ holds. □

Remark 4.6. Let (X_1, Q_1) and (X_2, Q_2) be two generating spaces of quasi-norm family (G.S.Q-N.F). If $\alpha \in (0,1)$ and $T : X_1 \to X_2$ is an α-level bounded linear operator, then
$|T(x)|^2_\alpha \leq |T|^s_\alpha |x|^1_{1-\alpha} \ \ \forall\ x \in X_1$.

Theorem 4.7. *Let $(X,\ Q)$ be a generating space of semi-norm family and $\alpha \in (0,1)$. If f is an α-level bounded linear operator which is defined on a subspace Z of X, then f has a linear extension $\hat{f}_\alpha$ from Z to X which is α-level bounded on X and $|f|^s_\alpha = |\hat{f}|^s_\alpha\ \forall \alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$ and we define
$p_{1-\alpha}(x) = |f|^s_\alpha \; |x|_{1-\alpha} \quad \forall \, x \in X.$
Then clearly $p_{1-\alpha}$ is a sublinear functional on X and $|f(x)| \leq p_{1-\alpha}(x) \;\; \forall \, x \in Z$.
So there exists a linear extension $\hat{f_\alpha}$ of f from Z to X satisfying
$|\hat{f_\alpha}(x)| \leq p_{1-\alpha}(x) \;\; \forall \, x \in X.$
$\Rightarrow \; |\hat{f_\alpha}(x)| \leq |f|^s_\alpha \; |x|_{1-\alpha} \;\; \forall \, x \in X.$
Hence $\hat{f_\alpha}$ is α-level bounded on X.
Again $|\hat{f_\alpha}|^s_\alpha = \bigvee\limits_{x \in X, |x|_{1-\alpha} \leq 1} \{|\hat{f_\alpha}(x)|_\alpha\} \leq |f|^s_\alpha \;\; \forall \, x \in X.$
Again from definition $|\hat{f_\alpha}|^s_\alpha \geq |f|^s_\alpha$. Hence $|\hat{f_\alpha}|^s_\alpha = |f|^s_\alpha \;\; \forall \alpha \in (0,1)$. □

Remark 4.8. If (X, Q) be a G.S.Q-N.F, then the Dual space $B(X, Q^s)$ of (X,Q) is a generating space of norm family.

Theorem 4.9. *Let (X, Q) be a generating space of semi-norm family and $x_0 \in X$ such that $|x_0|_{1-\alpha} \neq 0$ for some $\alpha \in (0,1)$. Then there exists an α-level bounded linear functional $\hat{f_\alpha}$ on X such that*
$|\hat{f_\alpha}|^s_\alpha = 1$ *and* $\hat{f_\alpha}(x_0) = |x_0|_{1-\alpha}$.

Proof. We consider the subspace Z of X consisting of all elements $x = cx_0$ where c is a scalar. On Z we define a linear functional f by
$f(x) = f(cx_0) = c|x_0|_{1-\alpha}$.
Then f is α-level bounded since $|f(x)| = |c||x_0|_{1-\alpha} = |cx_0|_{1-\alpha} = |x|_{1-\alpha}$ and $|f|^s_\alpha = 1$.
By theorem 4.7, f has a linear extension $\hat{f_\alpha}$ from Z to X with $|\hat{f_\alpha}|^s_\alpha = |f|^s_\alpha = 1$ and $\hat{f_\alpha}(x_0) = f(x_0) = |x_0|_{1-\alpha}$. □

Corollary 4.10. *For every $x \in X$ we have*
$$|x|_{1-\alpha} = \bigvee_{|f|^s_\alpha \neq 0, \; f \in B(X,Q)} \frac{|f(x)|}{|f|^s_\alpha}.$$
Hence if x is such that $f(x) = \theta$ for all $f \in B(X,Q)$, then $x = \theta$.

Proof. From the above theorem we have, writing x for x_0,
$$\bigvee_{|f|^s_\alpha \neq 0, \; f \in B(X,Q)} \frac{|f(x)|}{|f|^s_\alpha} \geq \frac{\hat{f_\alpha}(x)}{|\hat{f_\alpha}|^s_\alpha} = |x|_{1-\alpha}$$
and from $|f(x)| \leq |f|^s_\alpha |x|_{1-\alpha}$ we have
$$\bigvee_{|f|^s_\alpha \neq 0, \; f \in B(X,Q)} \frac{|f(x)|}{|f|^s_\alpha} \leq |x|_{1-\alpha}.$$
Hence proved. □

Theorem 4.11. *Let (X, Q) be a generating space of semi-norm family and $\alpha \in (0, 1)$. If $x_0 \in X$ be any point on the surface of the sphere $\{x : |x|_{1-\alpha} \leq r (\neq 0)\}$, i.e. $|x_0|_{1-\alpha} = r$, then there exists a supporting hyperplane to the sphere $\{x : |x|_{1-\alpha} \leq r\}$ at the point x_0.*

Proof. The equation of the supporting hyperplane for the sphere $\{x : |x|_{1-\alpha} \leq r\}$ is of the form

$\{x \in X : f(x) = r\,|f|^s_\alpha\}$, for some α-level bounded linear operator $f(\neq O)$.
By Theorem 4.9, there exists an α-level bounded linear operator f_0 such that $f_0(x_0) = |x_0|_{1-\alpha} = r$ and $|f_0|^s_\alpha = 1$.
Therefore $H = \{x \in X : f_0(x) = r\,|f_0|^s_\alpha\} = \{x \in X : f_0(x) = r\}$ is a supporting hyperplane to the sphere $\{x : |x|_{1-\alpha} \leq r\}$.
Since $f_0(x_0) = |x_0|_{1-\alpha} = r$, it follows that the supporting hyperplane H passes through x_0. □

Theorem 4.12. *Let (X, Q) be a generating space of semi-norm family. Let $y_0 \in X - Z$.*
Let $d_\alpha = \bigwedge_{x \in Z} |y_0 - x|_{1-\alpha} > 0$ for some $\alpha \in (0, 1)$, then there exists an α-level bounded linear functional f_α on X such that
1) $f_\alpha(x) = 0 \ \ \forall\, x \in Z$,
2) $f_\alpha(y_0) = 1$,
3) $|f_\alpha|^s_\alpha = \frac{1}{d_\alpha}$.

Proof. The subspace $\{Z + y_0\}$ is uniquely representable in the form $y = x + ty_0$ where $x \in Z$ and t is real.
Let us define a functional ϕ_α on $\{Z + y_0\}$ by
$\phi_\alpha(y) = t$ for $y = x+ty_0 \in \{Z+y_0\}$. Then ϕ_α is a linear functional on $\{Z+y_0\}$. Also $\phi_\alpha(x) = 0 \ \ \forall\, x \in Z$ and $\phi_\alpha(y_0) = 1$.
Now $|\phi_\alpha(y)| = |t| = \frac{|t||y|_{1-\alpha}}{|y|_{1-\alpha}}$
$= \frac{|ty|_{1-\alpha}}{|y|_{1-\alpha}} = \frac{|ty|_{1-\alpha}}{|x+ty_0|_{1-\alpha}}$
$= \frac{|y|_{1-\alpha}}{|\frac{x}{t}+y_0|_{1-\alpha}} = \frac{|y|_{1-\alpha}}{|y_0-(-\frac{x}{t})|_{1-\alpha}} \leq \frac{|y|_{1-\alpha}}{d_\alpha}$.
So ϕ_α is an α-level bounded linear functional on $\{Z + y_0\}$ and

$$|\phi_\alpha|^s_\alpha = \bigvee_{y \in \{Z+y_0\},\ |y|_{1-\alpha} \leq 1} \{|\phi(y)|\} \leq \frac{1}{d_\alpha} \text{.......(i)}$$

Since $d_\alpha = \bigwedge_{x \in Z} |y_0 - x|_{1-\alpha}$, there exists a sequence $\{x_n\}$ in Z such that
$\Rightarrow \lim_{n\to\infty} |x_n - y_0|_{1-\alpha} = d_\alpha$.
Now $|\frac{\phi_\alpha(x_n-y_0)}{|x_n-y_0|_{1-\alpha}}| \leq |\phi_\alpha|^s_\alpha$
$\Rightarrow |\phi_\alpha(x_n - y_0)| \leq |\phi_\alpha|^s_\alpha\, |x_n - y_0|_{1-\alpha}$.
But $|\phi_\alpha(x_n - y_0)| = |\phi_\alpha(x_n) - \phi_\alpha(y_0)| = 1$
$\Rightarrow |\phi_\alpha|^s_\alpha\, |x_n - y_0|_{1-\alpha} \geq 1$
$\Rightarrow \lim_{n\to\infty} |x_n - y_0|_{1-\alpha}|\phi_\alpha|^s_\alpha = d_\alpha|\phi_\alpha|^s_\alpha \geq 1$
$\Rightarrow \lim_{n\to\infty} |\phi_\alpha|^s_\alpha d_\alpha \geq 1$
$\Rightarrow |\phi_\alpha|^s_\alpha \geq \frac{1}{d_\alpha}$......(ii)
From (i) and (ii) we have, $|\phi_\alpha|^s_\alpha = \frac{1}{d_\alpha}$.
By Theorem 4.7, ϕ_α has a linear extension f_α from $\{Z+y_0\}$ to X which is an α-level bounded linear functional on X such that the conditions (1), (2) and (3) hold. □

5. Conclusion

In this paper, we try to give a constructive proof of Hahn-Banach extension theorem in finite dimensional generating spaces of quasi-norm family with full generality. We have seen that under certain conditions, it can be extended to a countably infinite dimensional spaces. On the other hand we establish Hahn-Banach extension theorem on generating spaces of semi-norm family and some consequences of the same theorem are studied. We think that there is a large scope of developing more results of functional analysis in this context.

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